Exact energy spectrum of a two-temperature kinetic Ising model

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We develop the exact energy spectrum for a two-temperature kinetic Ising spin chain and its dual reaction-diffusion system with spatially alternating pair annihilation and creation rates. We also discuss the symmetries of the system pseudo-Hamiltonian and their role in developing a general solution. The surprisingly simple form for the eigenvalues leads to interesting physical consequences and to a possible numerical analysis of the dynamical properties of the system.

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I. INTRODUCTION

Over the last three decades, an increasing number of condensed-matter theorists are devoting their efforts to understanding complex collective behavior of far-from-equilibrium systems using methods that range from easily accessible computer simulations to sophisticated theoretical studies. Although great progress has been made, a comprehensive theoretical framework is still lacking.

This paper presents the exact energy spectrum of two closely related one-dimensional nonequilibrium models: a kinetic Ising spin chain (KISC) with cells coupled alternately to one of two temperature baths, with generalized Glauber dynamics [1], and its dual counterpart, a reaction-diffusion system (RDS) with spatially alternating pair annihilation and creation rates. Interest in these models is motivated by their experimental applications. Multi-temperature spin systems are fairly common: nuclear magnetic resonance in an external magnetic field is an example; a lattice of nuclei in a solid prepared at a finite spin temperature [2] is another. On the other hand, the RDS model with spatially alternating annihilation and creation rates is known to describe the dynamics of photoexcited solitons in polymers [3]. Mobilia *et al.* proposed an experimental realization of the RDS model with alternating rates in compounds with alternating metal and halogen atoms (MX chain compounds) using a laser with spatially modulated power output [4].

The two-temperature kinetic Ising model (KISC) was first introduced by Rácz and Zia [5] who calculated exactly the two-point correlation functions for the steady state. Using a perturbation expansion of the master equation, Schmüser and Schmittmann [2] calculated the first two corrections to the equilibrium Boltzmann distribution. Mobilia *et al.* [4] found an analytical solution for the full dynamics (magnetization, particle density, and all correlation functions) of this nonequilibrium spin chain and its related reaction-diffusion model using a generating function approach. Time dependence of the approach to the steady state for small systems has been exhibited in recent work by Mazilu and Williams [6]. Outstanding challenges include knowledge of the exact energy spectrum of these models and a compact expression for their steady states.

Our study brings us one step closer to achieving this goal. Using the standard mapping [7] of reaction-diffusion models onto integrable quantum chains, the RDS model can be expressed in a "free fermion" form by defining a quadratic non-Hermitian "stochastic Hamiltonian" [4]. This operator can be diagonalized as long as certain constraints are obeyed [8]. In this paper, we derive the exact energy spectrum of this pseudo-Hamiltonian. The methodology we use to extract the spectrum also establishes means for numerical evaluation of eigenstates which can be the basis of more complete systematic future studies of this model. The method can be generalized to find the energy spectrum and the dynamical properties of other models such as reaction-diffusion models with biased diffusion and dimer creation-annihilation or multi-temperature kinetic systems.

Our paper is organized as follows: in Sec. II, we give an overview of the models. Next (Sec. III), we describe the symmetries exhibited by the pseudo-Hamiltonian operator and their role in the diagonalization process. In Sec. IV, following some standard technical steps (Jordan Wigner transformation, discrete Fourier transform, and a generalized Bogoliubov transformation) we derive closed-form expressions for the eigenvalues and a methodology for extracting the eigenvectors. Section V presents a summary of our results, physical implications of these results, and some possible generalizations of this model.

II. OVERVIEW OF MODEL

Two equivalent one-dimensional models motivate the work herein: the KISC and its associated RDS model. The KISC model parallels the one-dimensional Ising model. We postulate a lattice of N side-by-side cells, numbered $n=1,2,\ldots,N$, arranged in a ring such that cell $n=N$ is considered adjacent to cell $n=1$. N is restricted to even values. Each cell has a single degree of freedom with two possible values: -1, which can be thought a cell occupied by a particle with spin down, and $+1$, describing a cell occupied by a particle with spin up. Each cell interacts with its two nearest neighbors, as well as being in contact with a heat bath at one of two temperatures—T_e for even-numbered cells and T_o for odd-numbered cells. If $T_e \neq T_o$ the system cannot achieve equilibrium: each heat bath tries to drive the system toward a different equilibrium state. As a result, energy flows continuously between the even cell sublattice and the odd. Configu-

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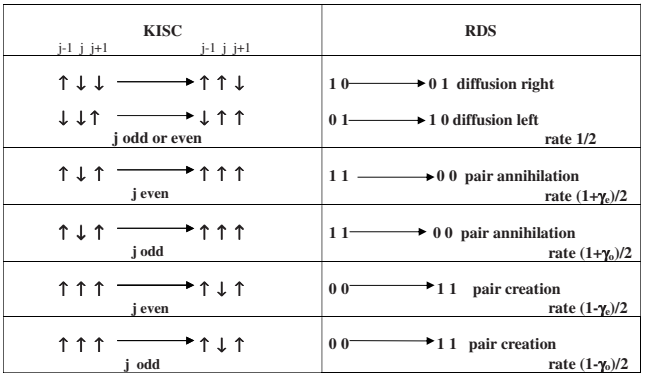

FIG. 1. Correspondence between the two-temperature KISC and the equivalent RDS model.

ration C (a list of the states of the N cells) changes into a different configuration C' with generalized Glauber transition rates $r[C \to C']$. Rate r is nonzero only if C and C' differ in the spin of a single particle. The rate at which site n has its spin flipped is given by

$$r_n = \frac{1}{2} - \frac{\gamma_n}{4} d_n (d_{n-1} + d_{n+1}), \tag{1}$$

where the factor $\gamma_n (0 \le \gamma_n \le 1)$ is related to the temperature of cell n by

$$\gamma_n = \begin{cases} \tanh\left(\dfrac{2}{k_B T_e}\right) & \text{for } n \text{ even} \\ \tanh\left(\dfrac{2}{k_B T_o}\right) & \text{for } n \text{ odd}, \end{cases} \tag{2}$$

k_B is Boltzmann's constant and d_n is the state (+1 or −1) of the nth cell. This rate equation prescribes a spin-flip rate for a cell of 1/2 if cells to the left and right have opposite spins, $(1-\gamma_n)/2$ if adjacent spins are the same and the same as that of cell n, and $(1+\gamma_n)/2$ if adjacent spins are the same and opposite that of cell n. The time scale is arbitrary.

The KISC model is mapped onto an equivalent reaction-diffusion model with spatially alternating pair creation and annihilation rates in the following way. A dual lattice of N sites is established, in which a site in the dual lattice is associated with the boundary between two sites in the KISC lattice. A pair of adjacent KISC spins with opposite signs is identified with a particle in the dual lattice; adjacent spins with the same sign are identified with the absence of a particle (a hole.) A spin flip in the KISC model translates into either diffusion of particles on the dual lattice with equal left-right rates or pair creation or annihilation with different rates. Transition rates between configurations in the KISC system become diffusion, pair creation and annihilation rates in the RDS system, as shown in Fig. 1.

Time evolution of these systems is described by the master equation expressing conservation of probability assuming a continuous-time dynamics. The probability $P(C,t)$ of finding the system in configuration C at time t increases due to transfer of probability into C from other configurations and decreases as C passes probability into others, in such a way that $\Sigma_C P(C,t) = 1$ for all t. The evolution of probability $P(C,t)$ is described by transition rates $r[C \to C']$, the probability per unit time that configuration C changes into a different configuration C'. The master equation is

$$\frac{dP(C,t)}{dt} = \sum_{C' \neq C} \{ r[C' \to C] P(C',t) - r[C \to C'] P(C,t) \} \tag{3}$$

in which the first term on the right represents the gain in probability of configuration C due to transitions from other configurations, and the second represents losses due to C transforming into other configurations.

We utilize Dirac notation to represent each configuration as $|C\rangle$. From this we build a vector representation of a probabilistic superposition of all possible configurations of a system:

$$|P(t)\rangle = \sum_C P(C,t) |C\rangle. \tag{4}$$

The master equation can now be re-expressed in terms of this vector as

$$\frac{d}{dt} |P(t)\rangle = -H |P(t)\rangle, \tag{5}$$

where the pseudo-Hamiltonian H is a $2^N \times 2^N$ matrix, with matrix elements

$$\langle C' | H | C \rangle = -r(C \to C'), \quad C' \neq C, \tag{6}$$

$$\langle C | H | C \rangle = \sum_{C' \neq C} r(C \to C'). \tag{7}$$

A formal solution to Eq. (5) can be written as $|P(t)\rangle = e^{-Ht} |P(0)\rangle$. Our goal is to investigate the eigenvalues of operator H in order to explain the system's time dependence. From this point, we shall focus on the RDS model. We follow the precedent of representing the "particles" and "holes" in the dual lattice by a spin-1/2 model: a particle is represented by spin up ($|1\rangle$); a hole becomes spin down ($|0\rangle$). From the above formalism comes the definition of the probability-conserving operator H that controls the system's time dependence:

$$\begin{aligned} -2H = & \sum_{j \ even} [\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ + (1+\gamma_e) \sigma_j^+ \sigma_{j+1}^+ \\ & + (1-\gamma_e) \sigma_j^- \sigma_{j+1}^- - \gamma_e (\sigma_j^- \sigma_j^+ + \sigma_{j+1}^- \sigma_{j+1}^+) - (1-\gamma_e)] \\ & + \sum_{j \ odd} [\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ + (1+\gamma_o) \sigma_j^+ \sigma_{j+1}^+ \\ & + (1-\gamma_o) \sigma_j^- \sigma_{j+1}^- - \gamma_o (\sigma_j^- \sigma_j^+ + \sigma_{j+1}^- \sigma_{j+1}^+) - (1-\gamma_o)]. \end{aligned} \tag{8}$$

The operators σ_n^+ and σ_n^- are the Pauli spin raising and lowering operators on the nth cell:

$$\sigma_n^+ |0\rangle_n = |1\rangle_n, \quad \sigma_n^+ |1\rangle_n = 0, \quad \sigma_n^- |0\rangle_n = 0, \quad \text{and} \quad \sigma_n^- |1\rangle_n = |0\rangle_n.$$

It has been shown that the eigenvalues and eigenvectors of H can be found if the "free fermion constraint" is

obeyed [8]. For the RDS model, this means that the sum of local diffusion rates is equal to the sum of local pair creation and annihilation rates [in our case $(1/2)+(1/2)=(1+\gamma_{o,e}/2)+(1-\gamma_{o,e}/2)$]. This constraint assures the bilinearity of the H operator, and, consequently, an exact solution for the problem.

III. SYMMETRIES OF THE H OPERATOR

Symmetries exhibited by the H operator [Eq. (8)] affect the form of its eigenvalues and eigenvectors and in some cases aid in the process of determining them. In this section we discuss these symmetries and indicate their role in the eigenvalue analysis to follow.

Because the fundamental process described by the rate equation, Eq. (1), corresponds to the simultaneous flipping of two spins in the RDS model, the H operator does not change the "spin-parity" of a state; i.e., states with an even number of up spins are transformed by H into states of only even number of up spins and likewise for odd numbers of up spins. This symmetry immediately separates the configuration space into two subspaces of the same dimensionality $(2^N/2)$ which do not interact. Thus, the $2^N \times 2^N$ H matrix is reduced to two equal-sized diagonal blocks by proper ordering of the configuration basis states. This reduction is central to our derivation of results for arbitrary N.

The H matrix is also invariant to a translation of the ring of cells by an even number of cells to the right or left. Thus, H commutes with the operator that invokes this translation and simultaneous eigenstates of the two operators can be found. Such eigenstates are conveniently written as sums of the form

$$|\mathcal{C}_p\rangle = \sum_{n=0}^{N/2-1} e^{inp(4\pi/N)}|C_n\rangle, \tag{9}$$

$(p=0,\ldots,N/2-1)$, where $|C_0\rangle$ is a spin configuration and $|C_n\rangle$ is the same configuration pushed $2n$ cells to the right using periodic boundary conditions (thus, if $|C_0\rangle=|011000\rangle$, then $|C_2\rangle=|100001\rangle$). The translation symmetry of the pseudo-Hamiltonian implies that H does not mix states of the form $|\mathcal{C}_p\rangle$ which have different p values. Thus, within each of the two major sub-blocks of the H matrix (one of even spin parity; one of odd spin parity) there are $N/2$ smaller sub-blocks, each with a different value of p. This symmetry has enabled relatively straightforward extraction of eigenvalues and eigenstates for even N values up to $N=8$ (with a 256-dimensional configuration space) [9]. It also motivates the form of the discrete Fourier transform used in the following section, where the case for arbitrary N is considered.

Two additional symmetries are apparent from the form of H given above that provide further information regarding the form of the eigenvalues and eigenvectors. The simultaneous translation of the spin chain by a single site (cell n becomes cell $n+1$) along with the interchange of values $\gamma_e \leftrightarrow \gamma_o$ leaves H invariant. If we use $\mathcal{X}$ to represent this transformation, it follows that $\mathcal{X}^2=I$. If $|\psi\rangle$ is an eigenstate of H with eigenvalue E, then

$$\mathcal{X}H|\psi\rangle = H\mathcal{X}|\psi\rangle = \mathcal{X}E\mathcal{X}^{-1}\mathcal{X}|\psi\rangle.$$

This leads to several possibilities:

(i) E is invariant under the interchange $\gamma_e \leftrightarrow \gamma_o$ and either $\mathcal{X}|\psi\rangle$ is a constant multiple of $|\psi\rangle$ or $\mathcal{X}|\psi\rangle$ produces another eigenstate of H distinct from $|\psi\rangle$ but with the same eigenvalue.

(ii) E is not invariant under the interchange $\gamma_e \leftrightarrow \gamma_o$, but instead transforms to another distinct eigenvalue of H, and $\mathcal{X}|\psi\rangle$ becomes a corresponding eigenstate.

Explicit diagonalization for small N has suggested that the eigenvalues are invariant under $\gamma_e \leftrightarrow \gamma_o$. The general solution for the eigenvalues presented below shows this to be true for all even values of N.

Another symmetry operation leaving H invariant consists in changing the sign of both γ constants and simultaneously flipping every spin. If we use $\mathcal{X}_A$ to represent this transformation, algebra similar to that of the prior paragraph leads to the following:

(i) E is invariant under the change in sign of both γ's and either $\mathcal{X}_A|\psi\rangle$ is a constant multiple of $|\psi\rangle$ or $\mathcal{X}_A|\psi\rangle$ produces another eigenstate of H distinct from $|\psi\rangle$ but with the same eigenvalue.

(ii) E is not invariant under the change in sign of both γ's, but instead transforms to another distinct eigenvalue of H, and $\mathcal{X}_A|\psi\rangle$ becomes a corresponding eigenstate.

Results of explicit diagonalization for small N have also suggested that the eigenvalues are invariant under the change in sign of both γ's. This proves also to be true for all N.

Results from explicit diagonalization for systems with N up to 8, enabled by utilization of symmetries of H, reveal the following regularities, which will be seen to be features of the general solution: (i) there are exactly two 0 eigenvalue states, one in the even and one in the odd spin-parity subspace; (ii) the maximum eigenvalue in both the even and odd spin-parity blocks has value N; (iii) each 0 eigenvalue state is in the $p=0$ subspace, with eigenvectors containing symmetric sums of shifted states; (iv) eigenvalues depend only upon the single parameter $\sqrt{\gamma_e\gamma_o}$. This implies that the spectrum can be fully deduced through knowledge of the spectrum for the single-temperature case, $T_e=T_o$.

IV. GENERAL CASE

Here we approach the problem for general N values using standard methodology [8]. Starting with the full, two-temperature operator in terms of spin raising and lowering operators [Eq. (8)] we can rewrite in terms of fermionic operators $(c_j, c_j^\dagger)$ that satisfy anticommutation relations

$$\{c_j, c_i^\dagger\} \equiv c_j c_i^\dagger + c_i^\dagger c_j = \delta_{j,i},$$

$$\{c_j, c_i\} = \{c_j^\dagger, c_i^\dagger\} = 0.$$

This transformation, due to Jordan and Wigner [10], is as follows:

$$\sigma_j^+ = c_j^\dagger \exp\left(i\pi \sum_{i<j} c_i^\dagger c_i\right),$$

$$\sigma_j^- = c_j \exp\left(-i\pi \sum_{i<j} c_i^\dagger c_i\right).$$

Straightforward application of this transformation produces the "fermionized" pseudo-Hamiltonian

$$\begin{aligned} H = &-\frac{1}{2}\sum_{j\ even} [c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j + (1+\gamma_e)c_j^\dagger c_{j+1}^\dagger \\ &- (1-\gamma_e)c_j c_{j+1} + \gamma_e(c_j^\dagger c_j + c_{j+1}^\dagger c_{j+1}) - (1-\gamma_e)] \\ &-\frac{1}{2}\sum_{j\ odd} [c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j + (1+\gamma_o)c_j^\dagger c_{j+1}^\dagger - (1-\gamma_o)c_j c_{j+1} \\ &+ \gamma_o(c_j^\dagger c_j + c_{j+1}^\dagger c_{j+1}) - (1-\gamma_o)]. \end{aligned} \tag{10}$$

For the even spin-parity subspace the requirement that $\sigma_{N+1}^+ = \sigma_1^+$ and $\sigma_{N+1}^- = \sigma_1^-$ implies $c_{N+1} = -c_1$ and $c_{N+1}^\dagger = -c_1^\dagger$. For the odd spin-parity case, $c_{N+1} = c_1$ and $c_{N+1}^\dagger = c_1^\dagger$. We take advantage of the translation symmetry described in Sec. III by defining two kinds of fermions in momentum space, one created from even-numbered cells, the other from odd-numbered cells. Define the following momentum-space operators:

$$a_q^\dagger = e^{i\pi/4}\sqrt{\frac{2}{N}}\sum_{j\ even} c_j^\dagger e^{i(j/2)q}, \quad c_j^\dagger = e^{-i\pi/4}\sqrt{\frac{2}{N}}\sum_{q\in Q} a_q^\dagger e^{-i(j/2)q},$$

$$b_q^\dagger = e^{i\pi/4}\sqrt{\frac{2}{N}}\sum_{j\ odd} c_j^\dagger e^{i((j+1)/2)q},$$

$$c_j^\dagger = e^{-i\pi/4}\sqrt{\frac{2}{N}}\sum_{q\in Q} b_q^\dagger e^{i((j+1)/2)q}.$$

We chose q values to belong to the set $Q = \{\pm\frac{2\pi}{N}, \pm\frac{6\pi}{N}, \pm\frac{10\pi}{N}, \ldots, \pm\frac{(N-2)\pi}{N}\}$ for states with even spin-parity, and $Q = \{0, \pm\frac{4\pi}{N}, \pm\frac{8\pi}{N}, \ldots, \pm\frac{(N-4)\pi}{N}, \pi\}$ for states with odd spin-parity, to assure proper periodic boundary conditions for each case. These definitions of Q assume that $N/4$ is integer valued, but the ultimate results are valid as long as N is even.

Like the operators c_j and $c_j^\dagger$, the momentum-space operators obey the canonical fermionic anticommutation relationships:

$$\{a_q, a_{q'}\} = \{a_q^\dagger, a_{q'}^\dagger\} = 0, \quad \{a_q, a_{q'}^\dagger\} = \delta_{q,q'},$$

$$\{b_q, b_{q'}\} = \{b_q^\dagger, b_{q'}^\dagger\} = 0, \quad \{b_q, b_{q'}^\dagger\} = \delta_{q,q'},$$

$$\{a_q, b_{q'}\} = \{a_q^\dagger, b_{q'}^\dagger\} = 0, \quad \{a_q, b_{q'}^\dagger\} = 0.$$

In terms of these operators, the pseudo-Hamiltonian is written as

$$\begin{aligned} H = \sum_{q\in Q} \Bigg[&-\cos\left(\frac{q}{2}\right)(e^{i(q/2)}a_q^\dagger b_q + e^{-i(q/2)}b_q^\dagger a_q) \\ &+ a_q^\dagger b_{-q}^\dagger \frac{i}{2}[(1+\gamma_e)e^{iq} - (1+\gamma_o)] \\ &+ a_q b_{-q} \frac{i}{2}[(1-\gamma_e)e^{-iq} - (1-\gamma_o)] \\ &- \frac{(\gamma_e+\gamma_o)}{2}(a_q^\dagger a_q + b_q^\dagger b_q) + 2 + \gamma_e + \gamma_o \Bigg]. \end{aligned} \tag{11}$$

The ultimate step in the derivation is a Bogoliubov-type similarity transform to new variables in which H takes a diagonal form. Following the method of Lieb *et al.* [11] we begin by postulating a form for H that reads

$$H = \sum_q (\omega_q \tilde{\chi}_q \chi_q + \omega_q' \tilde{\xi}_q \xi_q + \text{const}) \tag{12}$$

using operators $\tilde{\chi}_q$, χ_q, $\tilde{\xi}_q$, and ξ_q which obey fermionic anticommutation relations:

$$\{\chi_q, \chi_{q'}\} = \{\tilde{\chi}_q, \tilde{\chi}_{q'}\} = 0, \quad \{\chi_q, \tilde{\chi}_{q'}\} = \delta_{q,q'},$$

$$\{\xi_q, \xi_{q'}\} = \{\tilde{\xi}_q, \tilde{\xi}_{q'}\} = 0, \quad \{\xi_q, \tilde{\xi}_{q'}\} = \delta_{q,q'},$$

$$\{\chi_q, \xi_{q'}\} = \{\tilde{\chi}_q, \tilde{\xi}_{q'}\} = 0, \quad \{\chi_q, \tilde{\xi}_{q'}\} = 0.$$

Since H is not Hermitian, $\tilde{\chi}_q \neq \chi_q^\dagger$ and $\tilde{\xi}_q \neq \xi_q^\dagger$.

From the form of Eq. (12) it is straightforward to evaluate the commutators

$$[\chi_q, H]_- = \omega_q \chi_q, \quad [\xi_q, H]_- = \omega_q' \xi_q \tag{13}$$

and

$$[\tilde{\chi}_q, H]_- = -\omega_q \chi_q, \quad [\tilde{\xi}_q, H]_- = -\omega_q' \xi_q. \tag{14}$$

We define the four operators in the χ_q, ξ_q basis in terms of the four a_q and b_q operators using a 4×4 matrix D:

$$\begin{pmatrix} \tilde{\chi}_q \\ \chi_{-q} \\ \tilde{\xi}_q \\ \xi_{-q} \end{pmatrix} = D \begin{pmatrix} a_q^\dagger \\ a_{-q} \\ b_q^\dagger \\ b_{-q} \end{pmatrix}. \tag{15}$$

The commutators in Eqs. (13) and (14) can be evaluated using Eq. (15) along with

$$[a_q^\dagger, H]_- = \gamma_{av} a_q^\dagger + \cos\frac{q}{2} e^{-Iq/2} b_q^\dagger + \epsilon' b_{-q},$$

$$[a_{-q}, H]_- = -\gamma_{av} a_{-q} + \bar{\epsilon} b_q^\dagger - \cos\frac{q}{2} e^{-Iq/2} b_{-q},$$

$$[b_q^\dagger, H]_- = \cos\frac{q}{2} e^{Iq/2} a_q^\dagger - \bar{\epsilon}' a_{-q} + \gamma_{av} b_q^\dagger,$$

$$[b_{-q}, H]_- = -\epsilon a_q^\dagger - \cos\frac{q}{2} e^{Iq/2} a_{-q} - \gamma_{av} b_{-q}, \tag{16}$$

with definitions

$$\gamma_{av} \equiv \frac{\gamma_e + \gamma_o}{2}, \quad \gamma_d \equiv \frac{\gamma_e - \gamma_o}{2},$$

$$\epsilon \equiv e^{iq/2}\left[-(1+\gamma_{av})\sin\frac{q}{2} + i\gamma_d \cos\frac{q}{2}\right],$$

$$\epsilon' \equiv e^{-iq/2}\left[(1-\gamma_{av})\sin\frac{q}{2} - i\gamma_d \cos\frac{q}{2}\right].$$

$\bar{\epsilon}$ represents ϵ as defined above but with q replaced by $-q$, and likewise for ϵ'.

The commutator equation $[\tilde{\chi}_q, H]_- = -\omega_q \tilde{\chi}_q$, using Eqs. (15) and (16), transforms into a linear equation in the independent operators $a_q^\dagger$, b_{-q}, $b_q^\dagger$, and a_{-q}. Coefficients of each of these four operators on the right-hand side must equal the corresponding coefficient on the left yielding a linear set of equations for the elements of the top row D_1 of the matrix D which can be expressed by the matrix equation $MD_1 = -\omega_q D_1$ where

$$M = \begin{bmatrix} \gamma_{av} & 0 & \cos\frac{q}{2} e^{Iq/2} & -\epsilon \\ 0 & -\gamma_{av} & -\bar{\epsilon}' & -\cos\frac{q}{2} e^{Iq/2} \\ \cos\frac{q}{2} e^{-Iq/2} & \bar{\epsilon} & \gamma_{av} & 0 \\ \epsilon' & -\cos\frac{q}{2} e^{-Iq/2} & 0 & -\gamma_{av} \end{bmatrix}. \tag{17}$$

Similarly, the commutator equation $[\tilde{\xi}_q, H]_- = -\omega_q' \tilde{\xi}_q$ produces the matrix equation $MD_3 = -\omega_q' D_3$ involving the third row of the matrix D; $[\chi_{-q}, H]_- = \omega_{-q} \chi_{-q}$ produces the matrix equation $MD_2 = \omega_{-q} D_2$ involving the second row of D; and $[\xi_{-q}, H]_- = -\omega_{-q}' \xi_{-q}$ produces the matrix equation $MD_4 = \omega_{-q}' D_4$ involving the fourth row of the matrix D. For a particular value of q, M thus has two positive eigenvalues corresponding to the energy of excitations generated by $\tilde{\chi}_{-q}$ and $\tilde{\xi}_{-q}$, and two redundant negative eigenvalues with absolute values corresponding to energies for $\tilde{\chi}_q$ and $\tilde{\xi}_q$ excitations.

Straightforward calculations produce eigenvalues for M of $\pm(1 \pm \cos\frac{q}{2}\sqrt{\gamma_e \gamma_o})$. We chose to identify $\omega_q = 1 + \cos\frac{q}{2}\sqrt{\gamma_e \gamma_o}$ and $\omega_q' = 1 - \cos\frac{q}{2}\sqrt{\gamma_e \gamma_o}$ in order to deduce the exact spectrum for the model. Corresponding eigenvectors can be extracted as well but their algebraic expressions are too complicated to be useful in analytic form.

There is a unique even spin-parity vacuum state $|0\rangle_e$ defined by the relations

$$\chi_q |0\rangle_e = 0, \quad \xi_q |0\rangle_e = 0$$

for

$$q \in \left\{ \pm\frac{2\pi}{N}, \pm\frac{6\pi}{N}, \pm\frac{10\pi}{N}, \ldots, \pm\frac{(N-2)\pi}{N} \right\}.$$

Other even spin-parity states are formed by an even number of excitations of the $\tilde{\chi}_q$ or $\tilde{\xi}_q$ type: each excitation of the former type carries energy ω_q, and each of the latter type carries ω_q'. Since these excitations are fermionic, there cannot be two $\tilde{\chi}_q$ excitations with the same q nor can there be two $\tilde{\xi}_q$ excitations with the same q. The highest energy state has $N/2$ distinct $\tilde{\chi}_q$ excitations and $N/2$ distinct $\tilde{\xi}_q$ excitations and carries total energy N.

The odd spin-parity sector of the spectrum also has a unique vacuum:

$$\chi_q |0\rangle_o = 0, \quad \xi_q |0\rangle_o = 0$$

for

$$q \in \left\{ 0, \pm\frac{4\pi}{N}, \pm\frac{8\pi}{N}, \ldots, \pm\frac{(N-4)\pi}{N}, \pi \right\}.$$

Other odd-parity states are formed by an even number of excitations of the $\tilde{\chi}_q$ or $\tilde{\xi}_q$ type with q with values from the list appropriate to odd spin-parity. The highest energy odd spin-parity state has $N/2$ distinct $\tilde{\chi}_q$ excitations and $N/2$ distinct $\tilde{\xi}_q$ excitations and carries total energy N.

V. CONSEQUENCES AND CONCLUSIONS

It is surprising that a relatively simple set of eigenvalues emerges from the great algebraic complexity of the solution for arbitrary N of the two-temperature model considered herein. In contrast, the eigenvectors exhibited in [9] for $N=4$ show that even in this relatively simple case, the state vectors are algebraically difficult. In particular, the even and odd $E=0$ eigenstates that correspond to steady-state solutions are not easily characterized. In general the eigenstates depend separately upon the values of γ_e and γ_o. While the methodology employed in the previous section can, in principle, allow the extraction of the eigenvectors of H, the algebraic complexity of the eigenvectors of the matrix M suggests that such a straightforward exposition of them is not likely to be illuminating. A full numerical exploration of the eigenstates that would enable numerical evaluation of particle densities and correlation functions is beyond the scope of the current work. General relationships for these have been previously exhibited by Mobilia *et al.* [4].

The fact that the eigenvalues depend upon the single parameter $\sqrt{\gamma_e \gamma_o}$ allows some simple deductions regarding special cases of the two-temperature model. If one of the temperature baths has infinite temperature (e.g., $\gamma_e=0$), the eigenvalues are those of the case of a single-temperature model with infinite temperature. These eigenvalues are identical to those of the Glauber model [1], but the eigenvectors are of greater complexity. If one of the temperature baths has $T=0$ (e.g., $\gamma_e=1$), the energy spectrum becomes the same as for the one-temperature case for a temperature related to but not equal to that of the other bath. In the RDS language, this case corresponds to a system with pair creation prohibited, and pair annihilation at a rate of 1 for the even sites. In general, for every case with distinct temperatures T_e and T_o, there is a single temperature that will yield the same energy eigenvalues. The spectrum of energies for the single-temperature case follows from the work of Grynberg *et al.* [8] [for the special case $h=h'=\frac{1}{2}$, $\epsilon=(1+\gamma)/2$, and $\epsilon'=(1-\gamma)/2$] with $\gamma=\tanh(2/k_BT)$.

A few general observations about the spectrum in the thermodynamic limit, $N\to\infty$, are possible. Paralleling an observation of Grynberg *et al.* [8] for the single-temperature case, as long as $\sqrt{\gamma_e \gamma_o}<1$ (at least one temperature bath is above absolute zero), there is a gap between the ground state and the next-highest energy level of $2(1-\sqrt{\gamma_e \gamma_o})$. This assures that states other than the steady state decay exponentially in time. The spectrum of remaining states consist of bands of energy levels centering on states with $E=4,6,8,\ldots$. The widths of these bands grow with E, while the spacing between adjacent states remains constant at 2. As a result, regardless of how small the parameter $\sqrt{\gamma_e \gamma_o}$ is, the bands will overlap for high energies. These observations are separately true for the even spin-parity and odd spin-parity segments of the energy spectrum.

Because H does not cause transitions between states of different spin-parity, the time evolution of any initial state can be broken into two independent segments. Any initial configuration of spin states can be broken into a piece with even spin-parity with probability P_e and one of odd spin-parity with probability P_o, with $P_e+P_o=1$. The even spin-parity segment decays toward the steady state $|0\rangle_e$ maintaining constant probability P_e; likewise, the odd segment decays toward $|0\rangle_o$ maintaining constant probability P_o. The separate even and odd spin-parity energies control the rate of decay of the non-steady-state components for each segment.

Symmetries discussed in Sec. II enable a few additional comments about the general form of the eigenvectors. Because we know the eigenvalues to be invariant under the interchange $\gamma_e \leftrightarrow \gamma_o$ it follows that displacement of the ring by a single site (effectively a permutation of basis states) along with an interchange of values of γ_e and γ_o should transform any eigenvector into a constant multiple of itself or into a different eigenvector with the same eigenvalue. Since the ground state ($E=0$) and the maximum energy state ($E=N$) for the even spin-parity sector and for the odd spin-parity sector are nondegenerate, each should be invariant within a constant under this transformation.

The eigenvalues are also invariant under the simultaneous change in sign of γ_e and γ_o. This implies that a simultaneous flip of all spins (another basis state permutation) accompanied by a sign change in both γs should transform an eigenvector into a constant multiple of itself or into another eigenvector with the same E. The nondegenerate eigenstates in each spin-parity sector should transform into constant multiples of themselves under this symmetry transformation.

Mobilia *et al.* [4] have examined the behavior of this model in the case where the γ's have opposite signs. Although the concept of negative temperatures does not make physical sense for the KISC model, in the context of the RDS system it corresponds to a grid where the pair creation rate exceeds the pair annihilation rate on one sublattice, and the opposite is true for the other sublattice. The eigenvalues for this case will have positive real parts $[\mathrm{Re}(E)=0,2,4,\ldots,N]$ corresponding to exponential damping in time, and imaginary parts proportional to $\sqrt{|\gamma_e \gamma_o|}$ producing oscillatory behavior. This result is consistent with the predictions presented in Mobilia *et al.* [4], for example, that under these conditions the density of particles approaches its equilibrium value via a term proportional to $\exp(-2t)\sin(2\sqrt{|\gamma_e \gamma_o|}t+\delta)$. Our spectrum results show such a behavior for the highest frequency oscillation associated with the most slowly decaying component of the state function evolving from a general initial condition.

In ongoing work we seek a compact expression for the steady state of these models. Given the relationship between spin systems and reaction-diffusion systems, it will be interesting to investigate the effect of various initial conditions and open boundary conditions on the dynamics of the system. From an experimental point of view, open boundary conditions for RDS systems would be important in the study of chemical reactions that include creation and annihilation processes and dimer deposition. Although particle densities and correlation functions can be calculated fairly straightforwardly in the thermodynamic limit, finite-size effects may also be worth investigating.

We can also imagine other extensions of the models presented. For example, we are interested in considering an RDS model with nonuniform diffusion rates for the odd and even sites and different creation and annihilation rates. This can also shed some light on the general problem of dimerized spin chains [12].

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